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Social network reciprocity as a phase transition in evolutionary cooperation

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In evolutionary dynamics the understanding of cooperative phenomena in natural and social systems has been the subject of intense research during decades. We focus attention here on the so-called "lattice reciprocity" mechanisms that enhance evolutionary survival of the cooperative phenotype in the prisoner's dilemma game when the population of Darwinian replicators interact through a fixed network of social contacts. Exact results on a "dipole model" are presented, along with a mean-field analysis as well as results from extensive numerical Monte Carlo simulations. The theoretical framework used is that of standard statistical mechanics of macroscopic systems, but with no energy considerations. We illustrate the power of this perspective on social modeling, by consistently interpreting the onset of lattice reciprocity as a thermodynamical phase transition that, moreover, cannot be captured by a purely mean-field approach.

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I. INTRODUCTION

Is the term "social temperature" just a rhetorical figure (suggestive metaphor), or on the contrary, could it be given a precise meaning? By working out in detail the evolutionary dynamics of the most studied social dilemma (the prisoner's dilemma) on a simple kind of artificial social network we will show here that the formal framework of equilibrium statistical mechanics is, to a large extent, applicable to the rigorous description of the asymptotic behavior of strategic evolution, thus providing the key for a formal quantitative meaning of the term social "temperature" in these contexts.

Evolutionary game theory, in contrast with classical game theory that focusses on the decision making process of (rational) agents, is concerned with entire populations of agents programmed to use some strategy in their interactions with other agents. The agents are replicators, i.e., entities which have the means of making copies of themselves (by inheritance, learning, infection, imitation, etc.), whose reproductive success depends on the payoff obtained during interaction. As the payoff depends on the current composition of strategies among the interacting agents, this yields a feedback loop that drives the evolution of the strategic state of the population [1–4].

This Darwinian feedback (frequency-dependent fitness) dynamics depends strongly not only on the particular game, and on the specifics of the way strategies spread, but also on the (social) structure of connections describing the interactions. Under the assumption of a well-mixed population (social panmixia assumption), the temporal evolution of the proportion of strategies among the population is governed by a differential equation called the replicator equation (see below). Well-known celebrated folk's theorems (see, e.g., Ref. [3]) establish a connection between the asymptotic behavior of this equation and the powerful concepts of classical game theory based on the notion of best reply (Nash). However, if the social panmixia assumption is abandoned, and individuals only interact with their neighbors in a social network, the asymptotic of evolutionary dynamics generically differ in a substantial way from this "well-mixed population" description. The social structure of strategic interactions turns out to be of importance regarding the evolutionary outcome of the strategic competition.

We will consider here the prisoner's dilemma (PD), a two-players-two-strategies game, where each player chooses one of the two available strategies, cooperation or defection: A cooperator receives R when playing with a cooperator and S when playing with a defector, while a defector earns P when playing with a defector and T (temptation) against a cooperator. When $T>R>P>S$, the game is a PD (while if $T>R>S>P$ it is called a snowdrift game or "chicken" or "hawks and doves"). Given the payoff's ordering, whatever the value of the prior assignment of probability to the co-player's strategy is, the expected payoff is higher for defection, and that is what a rational agent should choose. In the PD game only the defective strategy is a strict best response to itself and to cooperation, thus it is an easy example of game with an unbeatable [4] strategy. Still, though there is no difficulty in the making of the strategic decision from Nash analysis, two cooperators are better off than two defectors, hence the social dilemma.

In graph-structured populations, a large body of research (Refs. [5–18], and references therein) on evolutionary dynamics of the PD game has convincingly shown the so-called "lattice reciprocity" effects: The cooperative phenotype can take advantage of the topology of the social net, so that clusters of cooperators are often resilient to invasion by the (continuum-unbeatable) defective phenotype. This enhancement of asymptotic macroscopic levels of cooperation due to the structure and topology of strategic interactions includes, but it is far more general than the so-called space reciprocity mechanisms, where social nets are discretizations (solid state lattices) of Euclidian space and diffusion approximations are

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−0.019often useful [19]. In this regard, one should stress the accumulated evidence that (i) many interesting social nets [20–22] are far away from being regular lattices and (ii) freedom of connectivity scales (scale-free complex networks) enhances [23–27] the lattice reciprocity mechanisms up to unexpectedly high values of the temptation parameter T of the dilemma, where cooperation is very expensive (but affordable in an evolutionary sense).

In this paper we investigate in detail the lattice reciprocity mechanisms in an artificial network (dipole model) that models the competition for influence on a population of social PD imitators of two antagonist big brothers (nodes connected to the whole population but with no direct connection between them). The paper is organized as follows. The setting of evolutionary scenario [(a) game, (b) updating rule, and (c) network of social contacts] along with basic concepts and definitions, are given in Sec. II, where the dipole model is introduced. This is a closed system with a self-sustained evolutionary activity (nontrivial dynamics) of social cooperation, as we prove in Sec. III, where also the applicability of standard equilibrium statistical mechanics to the dipole model is assessed.

Explicit solutions of the evolutionary equilibrium probability measure of microstates are obtained in Sec. IV, for some special topologies of the fluctuating replicators subpopulation $\mathcal{F}$. There, we analyze first the two trivial limits for the graph structure of the target population $\mathcal{F}$, namely, complete graph and totally disconnected graph. One easily obtains an exact macroscopic (infinite size limit or thermodynamic limit) description for both cases by means of explicit differential equations for the macroscopic cooperation. A simple thermodynamical interpretation of the macroscopic behavior, is provided by the theorem of Sec. III, as we briefly outline. Next, for a simple "random regular graph" topology of $\mathcal{F}$, an explicit differential equation and simple thermodynamical predictions are obtained within a mean-field approximation in Sec. IV B. When compared to Monte Carlo numerical results, fundamental discrepancies are evident: while mean-field prediction does not show any critical behavior, our numerical results show beyond any doubt the existence of a thermodynamical phase transition at a critical value of the temptation T^*. This critical value separates apart two distinct equilibrium macroscopic phases of the fluctuating population, and signals the onset of macroscopic effects of lattice reciprocity. These effects are seen to operate as a positive feedback upon local fluctuations of the strategic neighborhoods, and thus they cannot be captured by "purely mean-field" macroscopic approaches. The concluding Sec. V tries to call interdisciplinary attention on the wide and utmost interesting prospectives for statistical physics "concepts and methods" in current studies on evolutionary dynamics and social systems modeling in general.

II. NATURAL STRATEGIC SELECTION ON GRAPHS

We specify here the evolutionary game dynamics scenario, meaning the game parametrization, the microscopic strategic dynamics (replication mechanism or strategic updating rule), and the social structure of contacts that we will consider along the paper. We normalize the PD payoffs to the reward for cooperating $R=1$ and fix the null payoff at punishment $P=0$. Note that provided the (differential or relative) selective advantage among two individuals depends on their payoff's difference (see below), one can arbitrarily fix the zero payoff level. Then only two parameters $T=b>1$ and $S=\epsilon\leqslant 0$ are tuned. Note that the range $\epsilon>0$ defines a game named "hawks and doves" (also "chicken" and "snowdrift") where punishment and sucker's payoff have the reverse order. We will occasionally comment on this range of parameters.

Moreover, we do not restrict our computations to $2R>T+S$. This restriction means that the total payoff for the two players is higher if both cooperate ($2R$) than if one cooperates and the other defects ($T+S$), and is usually incorporated in iterated games studies of the PD to prevent agents taking turns at defection and then sharing the payoffs. For the specifics of the replicator dynamics (memory-less, markovian) in the next paragraph, one should not expect that this restriction qualitatively matters.

Regarding the replication mechanism, we implement the finite population (size $N\gg 1$) analog of replicator dynamics [23,28]. At each time step t, which represents one generation of the discrete evolutionary time, each agent i plays once with each one of the agents in its neighborhood and accumulates the obtained payoffs P_i. Then, the individuals i synchronously update their strategies by picking up at random a neighbor j and comparing their respective payoffs P_i and P_j. If $P_i>P_j$, nothing happens and i keeps the same strategy for the next generation. On the contrary, if $P_j>P_i$, with probability $\Pi_{i\to j}=\eta(P_j-P_i)$, i adopts the strategy of its neighbor j for the next round robin with its neighbors, before which all payoffs are reset to zero. Here η is a number small enough to make $\Pi_{i\to j}$ an acceptable probability; its physical meaning is related to the characteristic inverse time scale: the larger it is, the faster evolution takes place.

From a theoretical point of view, this specific choice of the dynamics has the virtue of leading directly (see, e.g., Ref. [2]), under the hypothesis of a well-mixed population and very large population size, to the celebrated replicator equation for the frequencies p_α of strategies α(=C or D) in the population

$$\dot{p}_\alpha = p_\alpha(f_\alpha - \bar{f}), \tag{1}$$

where f_α, is the payoff of an α strategist and $\bar{f}$ is the average payoff for the whole population. Note that the time unit in Eq. (1) is scaled to η^{-1}.

For the payoffs of the prisoner's dilemma the asymptotic frequency of cooperators, from the replicator equation, is driven to extinction, $p_c=0$, while for the hawks and doves game, its asymptotic value is $\epsilon/(b-1+\epsilon)$. As stated in the introductory section, we will be concerned here mainly with populations that are not well-mixed, where predictions based on this nonlinear differential equation are often of little use.

Regarding the structure of connections between interacting agents, we will consider here that it is given by a fixed graph (i.e., connections between players do not change by rewiring) where agents are represented by nodes, and a link between nodes indicates that they interact (play). If k_i is the number of neighbors of agent i (connectivity or degree), and Δ is the maximal possible one-shot-payoff difference ($\Delta = \max\{b, b-\epsilon\}$), we will assume $\eta = (\max\{k_i, k_j\}\Delta)^{-1}$ for the specification of the probability $\Pi_{i\to j}$ of invasion of node i by the strategy of neighbor j. This simple choice, introduced in Ref. [23], assures that $\Pi_{i\to j} < 1$; in heterogeneous networks it also has the effect of slowing down the invasion processes from or to highly connected nodes, with respect to the rate of invasion processes between poorly connected nodes, a feature not without consequences [29].

We now introduce some notation, which is familiar to statistical physicists: The configuration (strategic microstate l) of a population of N agents at time t is specified by the sequence $l = \{s_i(t)\}$ ($i = 1, \ldots, N$), where $s_i(t) = 1$ (or 0) denotes that node i is at this time a cooperator (respectively, defector). The set of all possible 2^N configurations is called the phase space. Stationary probability densities of microstates $\mathcal{P}(l)$ ($l = 1, \ldots, 2^N$) are then representatives of strategic macrostates. The average cooperation c_l of microstate l is defined as

$$c_l = \frac{1}{N}\sum_{i}^{N} s_i. \qquad (2)$$

We denote by $\Pi_{l'l}$ the probability that the strategic microstate of the population at time $t+1$ is l', provided that it is l at time t. Note that $\Sigma_{l'}\Pi_{l'l} = 1$. A microstate $\hat{l}$ is a frozen equilibrium configuration if the probability that it changes in one time step is null, and then $\Pi_{\hat{l}\hat{l}} = 1$ and $\Pi_{l'\hat{l}} = 0$ if $l' \neq \hat{l}$. We will assume generic real values (irrational) of the payoff parameters, so that if a configuration contains a C-D link it cannot be a frozen configuration. The only possible frozen equilibrium configurations are all-C and all-D. However, for a very wide class of graphs, and a wide range of model parameters they are not the only possible stationary probability measures.

We now illustrate by means of easy examples the evolution of PD on graphs. Our first and simplest example is a star-shaped graph consisting of a central node connected to $N-1$ peripheral nodes. It is straightforward to check that any initial condition with cooperators at the central node and (at least) at $\{[b-\epsilon(N-1)]/(1-\epsilon)\}+1$ peripheral nodes has a positive probability of evolving in one time step to a configuration with a higher number of cooperators, and a null probability of evolving towards less cooperators. Thus, all those configurations evolve asymptotically to the all-C equilibrium. The rest of the configurations evolve towards the all-D equilibrium. Therefore, if $N > (b-\epsilon+2)$ both equilibria are attractors (absorbing states), in the sense that some configurations different from themselves evolve to them; the phase space is partitioned into two basins of attraction. If $N < b-\epsilon+2$, only the all-D frozen equilibrium is an attractor. The stationary probability densities $\mathcal{P}^*(l)$ of the star are pure point measures (two- or one-Dirac delta peaks) in the thermodynamic limit $N \to \infty$.

Now take a star and add some arbitrary number of links between its peripheral nodes. We call this network a crown, whose head is the central node. If the head is occupied at t_0 by a defector, it will remain so forever, because the payoff of a peripheral cooperator is strictly lower than head's payoff. Sooner or later the head (center) of the crown will be imitated by the whole crown, and the evolution will stop when everybody is defecting. But, what happens to a cooperator on the head? The answer is dependent on both, the net topology of the crown periphery and the cooperators disposition there: To ensure fixation of cooperation at the head node, it suffices that a subset C of peripheral nodes occupied by cooperators, and with no direct links to the rest of the periphery, have a size $n_C > bk_{\max} - \epsilon(N - n_C - 1)$, where $k_{\max}$ is the maximal degree in the rest of the periphery. Under this proviso all-C is the unique absorbing microstate of all corresponding initial conditions.

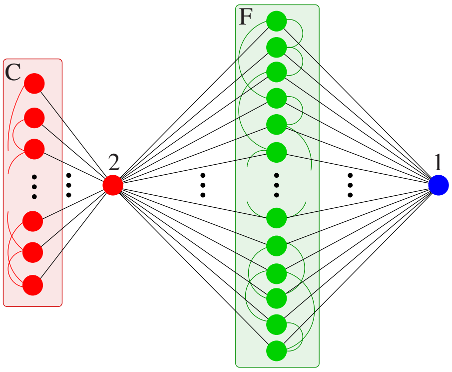

FIG. 1. (Color online) Structure of the dipole network. Two nodes (1 and 2) are connected to all nodes in $\mathcal{F}$, whose elements can be arbitrarily linked to each other. Moreover, node 2 is also linked to a set $\mathcal{C}$ (with arbitrary internal connections as well). Initial conditions are indicated by colors: red for cooperators (node 2 and set $\mathcal{C}$), blue for defectors (node 1), and green means arbitrary (set $\mathcal{F}$). See the text for further details.

Finally consider the graph schematized in Fig. 1, composed of the following. (a) A component $\mathcal{F}$ of n_F nodes with arbitrary connections among them. (b) A node, say node 1, that is connected to all the nodes in $\mathcal{F}$ and has no other links. (c) A component $\mathcal{C}$ of n_C nodes with arbitrary connections among them. (d) A node, say node 2, that is connected to all the nodes in $\mathcal{F}$ and $\mathcal{C}$, but not to node 1.

This is what we will call a dipole model network. It is a two-headed (nodes 1 and 2) crown (with periphery $\mathcal{F}$) plus a tail $\mathcal{C}$ hanging on head 2. To strengthen the special status of the head nodes, let us nickname them "big brothers." They certainly enjoy a sort of omnipresence that fits well with the character of Orwell's famous social science fiction novel *1984*. In the following section we prove that for this simple network there exists a nontrivial stationary probability density of microstates $\mathcal{P}^*(l)$ for the strategic evolution of the PD game.

III. THE DIPOLE MODEL

The analysis of evolutionary dynamics of the PD on the dipole network shows that there is a nontrivial invariant measure in phase space. Let us consider the set $\mathcal{I}$ of initial conditions defined by (i) big brother 1 is a defector, (ii) big brother 2 is a cooperator, and (iii) all nodes in component $\mathcal{C}$ are cooperators. Note that this set contains 2^{n_F} different configurations. We now prove that, provided some sufficient conditions, this is a minimally invariant set of the evolutionary dynamics.

First, one realizes that big brother 1 cannot be invaded by the cooperative strategy: The payoff of a cooperator node i in $\mathcal{F}$ is $P_i^c = k_i^c + 1 + \epsilon(k_i - k_i^c + 1)$, where k_i is the number of its neighbors in $\mathcal{F}$ and $k_i^c \leqslant k_i$ is the number of those that are cooperators. The payoff of big brother 1 (BB1) is then $P_1 \geqslant (k_i^c + 1)b$. For the PD game, where $\epsilon \leqslant 0$, the inequality $P_1 > P_i^c$ always holds, so that BB1 will always be a defector. [Note also that for the hawks and doves game, a sufficient condition for $P_1 > P_i^c$ is $b > 1 + \epsilon(k_F + 1)$, where k_F ($< n_F$) is the maximal degree in component $\mathcal{F}$, i.e., the maximal number of links that a node in $\mathcal{F}$ shares within $\mathcal{F}$.] We thus conclude that defection is fixed at BB1.

Second, thanks to its interaction with set $\mathcal{C}$, big brother 2 resists invasion, provided its size n_C is above a threshold: The payoff of a defector node i in $\mathcal{F}$ is $P_i^d = (k_i^c + 1)b$, where k_i^c is the number of its cooperator neighbors in $\mathcal{F}$, while the payoff of big brother 2 (BB2) is $P_2 = n_C + n_F\epsilon + n_F^c(1-\epsilon)$, where $n_F^c \leqslant n_F$ is the number of cooperators in $\mathcal{F}$. Thus, a sufficient condition for $P_2 > P_i^d$ is $n_C > b(k_F + 1) - n_F\epsilon$. With this proviso, BB2 will always be a cooperator, which in turn implies that all the nodes in the component $\mathcal{C}$ will remain always cooperators. Note that for $\epsilon \leqslant 0$ and $b > 1$, the absence of the component $\mathcal{C}$ could imply invasion of node 2, that would lead to fixation of the defective strategy on the whole network.

The previous argument proves that provided the sufficient conditions $n_C > b(k_F + 1) - \epsilon n_F$ and $b > 1 + \epsilon(k_F + 1)$ hold, the subset $\mathcal{I}$ of phase space defined by (i), (ii), and (iii) is an invariant set. As this set does not contain equilibria, no stochastic trajectory evolves from it to a frozen equilibrium configuration.

Finally, one realizes that $\mathcal{I}$ is indeed minimal, because at any time, a defector in $\mathcal{F}$ has a positive probability to be invaded by the cooperation strategy (at least from BB2), and a cooperator in $\mathcal{F}$ has a positive probability of being invaded by the defection strategy (at least from BB1). Therefore, any strategic configuration of the set $\mathcal{I}$ is reachable in one time step from any other, i.e., for all pairs (l, l') of microstates in $\mathcal{I}$, the transition probability $\Pi_{l'l} > 0$. Consequently, $\mathcal{I}$ does not contain proper invariant subsets: it is minimally invariant. Moreover, following Perron-Frobenius theorem, there exists a unique stationary macrostate $\mathcal{P}^*(l)$. This provides a rigorous framework for the interpretation of results from numerical Monte Carlo simulation studies in evolutionary dynamics on dipole models, provided the sufficient conditions above.

While nodes in $\mathcal{C}$ and big brother 2 are permanent cooperators, and big brother 1 is a permanent defector, nodes in $\mathcal{F}$ are forced to fluctuate. This partition of the network into sets of nodes where each particular strategy is fixed forever, and a set of fluctuating nodes, turns out to be a generic feature of the discrete replicator dynamics (neighbor imitation proportional to payoffs difference) on many network settings [24,26]. The simplicity of the dipole network model allows on it an easy formal proof of existence of this partition, so providing an illustration of both, its origins and generic character. It also shows the formal applicability of equilibrium statistical physics formalism to characterize the asymptotic behavior of evolutionary dynamics on these graphs. This will be made in the next section for specific choices of structural traits for the subgraph $\mathcal{F}$.

Let us note that if a direct link between BB1 and BB2 is added, then (see Appendix A) asymptotic fixation in the whole network of either cooperation or defection will occur, depending on the relative size n_C/n_F of components $\mathcal{C}$ and $\mathcal{F}$. The name "dipole" for this structure of connections is suggested by the strategic polar ($\mathcal{C}$-$\mathcal{F}$-$\mathcal{D}$) aspect of the whole graph. Note also that the number of $\mathcal{C}$-$\mathcal{F}$ and $\mathcal{F}$-$\mathcal{D}$ connections scales linearly with the size n_F of the fluctuating interior, that is to say that the poles (C and D) act as an externally imposed (AC) field on $\mathcal{F}$, whose strength is proportional to the internal levels of cooperation. As the cooperation (and then the fitness) levels are self-sustained (as proved by the previous theorem), this is a closed macroscopic system with a nontrivial self-sustained social activity of cooperation at evolutionary equilibrium.

The interest of the dipole model is by no means restricted to a mere academic illustration: First of all, we can make a technical use of it in macroscopic stability analysis studies of PD evolution on highly heterogeneous complex networks. Indeed, the fluctuations inside the subset $\mathcal{F}$ are the effect of the competition for invasion among two non-neighboring hubs (hugely connected nodes), where opposite pure strategies have reached fixation, in their common neighborhood. This is a local strategic configuration that mimics those that are often observed in stochastic simulations of evolutionary dynamics in highly heterogeneous (scale-free) networks [24,26]. Simple multipolar network models can easily be constructed (e.g., by establishing direct links from $\mathcal{C}$ to $\mathcal{F}$ in a way that simple sufficient conditions guarantee that the theorem still holds), that are indeed indistinguishable from typical strategic patterns found in the numerical simulations on scale-free networks. This makes the dipole net a very useful technical device to analyze the stability mechanisms of the cooperator clusters [24,26] in scale-free structured populations, as well as the kind of temporal fluctuations of cooperation that one should expect in the fluctuating set of nodes.

Regarding potentialities for econo-socio-physics applications of the dipole model, it could be viewed as a sort of schematic (then simplistic, cartoonlike) model for the competition for influence of two powerful superstructural institutions (e.g., such as "mass media," political parties, or lobbies) on a target population, in strongly polarized strategic contexts. The analysis rigorously provides sufficient conditions for the parameter values where fixation of strategic traits is proved impossible, so that temporal fluctuations dominate forever the target population of social imitators $\mathcal{F}$. The influence on each individual of the two competing institutions is simulated here through the omnipresent ("big brother" nodes 1 and 2) neighbors, whose own high appeal for imitation (the strength of big brother's influence) is in turn conditioned by the strategic composition of the target population. Here the interest could well be the study of the influence that metric and topological network characteristics of the social structure have on the strategic macrostate, and thus on the quantitative values of social indicators. We address some aspects of this issue in the next section. At a more general level, the design of experiments in social sciences as well as theoretical studies of artificial societies could greatly benefit from having at hand simple but nontrivial "exactly soluble statistical-mechanical models" that may provide safe guides to develop further intuitions on social phenomena that demand more comprehension.

IV. THE ROLE OF SOCIAL STRUCTURE IN BIG BROTHERS COMPETITION

In this section we present some analytical and numerical results on the evolutionary dynamics of games in the dipole model for different choices of topologies of the fluctuating set $\mathcal{F}$. The sufficient conditions stated in the previous section are assumed hereafter. We are interested in the situation where $n_F \gg 1$, i.e., large size of the fluctuating population.

First we will briefly comment on the straightforward limiting case when the macroscopic set $\mathcal{F}$ is a fully connected set, so that $k_F = n_F - 1$. This is the well-mixed population limit, for which it is easy to show that the replicator equation (1) is an exact description. The payoffs of polar nodes BB1 and BB2 are given by $P_1 = bcn_F$ and $P_2 = n_C + cn_F + \epsilon(1-c)n_F$, while the payoffs of a cooperator node and a defector node in $\mathcal{F}$ are $P_c = cn_F + \epsilon(n_F - cn_F + 1)$ and $P_d = (cn_F + 1)b$. One easily realizes that $P_c < P_d$, provided the sufficient condition ($b > 1 + \epsilon n_F$) for fixation of defection at node 1. Thus the (one time step) probabilities Q_{DC} (invasion of a cooperator node in $\mathcal{F}$) and Q_{CD} (invasion of a defector node in $\mathcal{F}$) are

$$Q_{DC} = \frac{1}{(n_F+1)} \frac{P_1 - P_c}{\Delta(n_F+1)} + \frac{(1-c)n_F}{(n_F+1)} \frac{P_d - P_c}{\Delta(n_F+1)},$$

$$Q_{CD} = \frac{1}{(n_F+1)} \frac{P_2 - P_d}{\Delta(n_F+n_C)}. \tag{3}$$

Assuming that the size of $\mathcal{F}$ is macroscopic, $n_F \gg 1$, the fraction of cooperators c in $\mathcal{F}$ evolves according to

$$\dot{c} = (1-c)Q_{CD} - cQ_{DC}. \tag{4}$$

Now, if $n_F \gg 1$, and $n_C/(n_F)^2 \to 0$, then both Q_{CD} and the first term on the right-hand side of Q_{DC} vanish, and we arrive at the differential equation

$$\dot{c} = \frac{c(1-c)}{\Delta}[\epsilon(1-c) - (b-1)c]. \tag{5}$$

This is, with a simple rescaling of time, the replicator equation (1): note that in the limit $n_F \gg 1$ that we have considered, the probability that a node in $\mathcal{F}$ picks up a big brother when updating its strategy is negligible, and then the evolution inside the complete graph $\mathcal{F}$ is overwhelmingly determined by the internal connections, and thus by the replicator equation. In other words, in this limit of maximal possible connectivity, BB1 and BB2 are no longer bigger than the nodes in $\mathcal{F}$ and their influence on the fluctuating set is negligibly small in the thermodynamic limit.

We now turn attention to situations where $k_F \ll n_F$, far from the social panmixia. In Sec. IV A we will explicitly solve the opposite trivial case of disconnected $\mathcal{F}$ set ($k_F = 0$), which turns out to reduce to the standard textbook ideal two-state model of statistical physics. After that, in Sec. IV B, the "random regular" network structure for $\mathcal{F}$ is seen to be amenable to a plausible mean-field approach, but insufficient to explain the phenomenology shown by Monte Carlo numerical results. These show beyond any doubt a critical behavior, a transition point separating two qualitatively different types of social macro-states. This transition is sensibly interpreted as the onset of lattice reciprocity. In other words, lattice reciprocity is a true critical social phenomenon.

A. $\mathcal{F}$ is a disconnected graph (ideal gas)

Let us now obtain some explicit results for one of the simplest choices for the topology of connections inside the fluctuating set, namely, $k_F = 0$. In this case each node in $\mathcal{F}$ is only connected to big brothers. This is in fact an effective single node problem, where homogeneity (i.e., mean field assumption) in $\mathcal{F}$ is exact; in other words, the absence of internal interactions in the set $\mathcal{F}$ is a sort of ideal-gas condition easy to exactly deal with in the large size limit.

Note that the sufficient conditions for fixation of defection at BB1 and of cooperation at BB2 are, respectively, $b > 1 + \epsilon$ and $n_C > b - \epsilon n_F$. Denoting by $c(t)$ the instantaneous fraction of cooperators in $\mathcal{F}$, one finds for the (one time step) probability Q_{DC} of invasion of a cooperator node in $\mathcal{F}$

$$Q_{DC} = \frac{cb - (1+\epsilon)/n_F}{2\Delta} \tag{6}$$

and using the notation $A = \epsilon + (n_C - b)/n_F$ and $B = 1 + n_C/n_F$

$$Q_{CD} = \frac{A + c(1-\epsilon)}{2\Delta B} \tag{7}$$

for the probability of invasion of a defector node in $\mathcal{F}$. Note that $A > 0$ due to the noninvasion of BB2 (sufficient) condition.

Provided $n_F \gg 1$, the fraction of cooperators c in $\mathcal{F}$ evolves according to the differential equation (4), which after insertion of expressions (6) and (7), and rescaling of time, becomes

$$\dot{c} = f(c) \equiv A_0 + A_1 c + A_2 c^2, \tag{8}$$

where the coefficients are

$$A_0 = A, \tag{9}$$

$$A_1 = 1 - \epsilon - A + B(1+\epsilon)/n_F, \tag{10}$$

$$A_2 = -(1 - \epsilon + bB). \tag{11}$$

One can easily check ($A_0 > 0$ and $A_2 < 0$) that there is always one positive root c^* of $f(c)$, which is the asymptotic value for any initial condition $0 \leq c(0) \leq 1$ of Eq. (8).

For $\epsilon = 0$, in the so called weak PD game (i.e., at the border between the PD and the hawks and doves game), if one further assumes that the relative size $\mu(F)$ of the component F is large enough, i.e., $\mu(F) \to 1$ and $\mu(C) \to 0$, one easily obtains that the stationary solution of Eq. (8) behaves as $c^* \simeq (b+1)^{-1}$ near the limit $\mu(F) \to 1$.

From the point of view of the set $\mathcal{F}$, when $n_F \gg 1$, the model corresponds to a noninteracting (ideal) set of independent phenotypic strategists that fluctuate due to a polar field (big brothers influence) whose strength is self-consistently determined by the average cooperation c. This problem is equivalent to the equilibrium of an ideal paramagnetic salt in a noisy (telegraphic) magnetic ac field of intensity proportional to the average magnetization.

A typical and correct statistical-physicists approach "from scratch" to this two-state model is the familiar microcanonical setting: At (dynamical) macroscopic equilibrium, the probability of each strategic microstate $l = \{s_i\}$ of fixed value of $c_l = c$ is uniform

$$P_l = \Omega^{-1}, \tag{12}$$

where $\Omega = n_F!/[(cn_F)!(n_F - cn_F)!]$ is their number. The lack of information $S = \ln \Omega$ of the macrostate as a function of global cooperation $n_F c$, i.e., the relation $S(n_F c)$, can be regarded as the analog of the microcanonical fundamental "thermodynamical" relation, and its first derivative is the intensive parameter β (thus the analog of the inverse thermodynamical temperature), that after using Stirling's approximation is easily obtained as

$$\beta = \ln\left(\frac{1-c}{c}\right). \tag{13}$$

This relation is the analog of a thermodynamical equation of state, which simply expresses the connection of the equilibrium value of the macroscopic cooperation level c to the "entropic" intensive parameter β. Note that c is determined by the balance condition ($\dot{c} = 0$)

$$\frac{1-c}{c} = \frac{Q_{DC}}{Q_{CD}}, \tag{14}$$

from where the equation of state (13) determines β as a function of model parameters (i.e., b, ϵ, and n_C/n_F). For example, when $\epsilon = 0$, $\beta = \ln b > 0$, indicating that the disorder of the activity increases with increasing cooperation. The maximal value of $\beta \to \infty$ corresponds to zero disorder ($b \to \infty$), while its minimal zero value corresponds to highest possible value (at $b = 1$) of cooperation $[c = (1/2)]$. Note that values of $b < 1$ correspond to negative β values, where entropy decreases with increasing values of cooperation, outside the PD domain ("stag hunt" game domain, see Ref. [30]).

An alternative (and equivalent in the thermodynamic limit) setting is to consider the whole space of 2^{n_F} configurations $l = \{s_i\}_{i=1}^{n_F}$, of unrestricted c_l, but under the condition that the average value $c = \Sigma_l \mathcal{P}_l c_l$ is fixed. This is the analog of the canonical setting. The normalization factor $Z = \Sigma_l \exp(-\beta c_l)$ is the analog of the familiar canonical partition function (Boltzmann's Zustandsumme), that due to the agents independence ($k = 0$) is easily factorized as $Z = [1 + \exp(-\beta)]^{n_F}$.

In the canonical setting a most informative macroscopic quantity is the "heat capacity" analog: The fluctuations of c_l along representative (typical) stochastic trajectories at equilibrium under the evolutionary dynamics of the game are, following the standard thermodynamical formalism, given by $\partial c / \partial(\beta^{-1})$, so that this quantitative social indicator detects very precisely sudden variations of the macroscopic cooperation with payoff's parameters. In this ideal-gas kind of case there are no critical points and fluctuations do not diverge. For example, for $\epsilon = 0$ they are given by the (Bernouillian) binomial variance $n_F c(1-c) = n_F b/(b+1)^2$.

B. $\mathcal{F}$ is a random regular graph

Random regular networks are random networks of fixed degree k. All nodes being thus equivalent, a sensible approach is to assume (mean-field-like, see, e.g., Ref. [31]) that the fraction of instantaneous cooperators in the neighborhood of a node is the fraction c of the whole set $\mathcal{F}$. In other words, one neglects local fluctuations of c. The contribution of the internal interactions to the variation of c is then of the "replicator equation" type, as discussed above for the complete graph case. The difference here is that if $k_F \ll n_F$ the contribution of the interactions with big brothers can no longer be neglected.

1. Mean-field approximation

The payoffs of big brothers BB1 and BB2 are given by $P_1 = bcn_F$ and $P_2 = n_C + cn_F + \epsilon(1-c)n_F$, while the payoffs of a cooperator node and a defector node at $\mathcal{F}$ under the mean-field assumption are

$$P_c = ck + 1 + \epsilon[k(1-c) + 1], \quad P_d = (ck+1)b. \tag{15}$$

The differential equation for c is then

$$\dot{c} = \frac{(1-c)(P_2 - P_d)}{(k+2)Bn_F\Delta} - \frac{c(P_1 - P_c)}{(k+2)n_F\Delta} + \frac{(1-c)ck(P_c - P_d)}{(k+2)^2\Delta}, \tag{16}$$

which under the assumption $kb \ll n_F$, takes the form

$$\dot{c} = f(c) \equiv \frac{1}{(k+2)^2 B\Delta}(A'_0 + A'_1 c + A'_2 c^2 + A'_3 c^3), \tag{17}$$

where the coefficients are

$$A'_0 = (k+2)(B - 1 + \epsilon), \tag{18}$$

$$A'_1 = 2[2(1-\epsilon) - B] + k[2(1-\epsilon) - B(b-\epsilon)] + k^2 B\epsilon, \tag{19}$$

$$A'_2 = 2(\epsilon - 1 - Bb) + k[\epsilon - 1 - B(1+\epsilon)] + k^2 B(1 - b - 2\epsilon), \tag{20}$$

$$A'_3 = k^2 B(b - 1 + \epsilon). \tag{21}$$

Note that the assumption $n_C > b - n_F\epsilon$ (i.e., the condition for big brother 2 to be a permanent cooperator) implies that $A'_0 > 0$, so that $\dot{c}(0) > 0$ and one positive root, say c^*, of $f(c)$

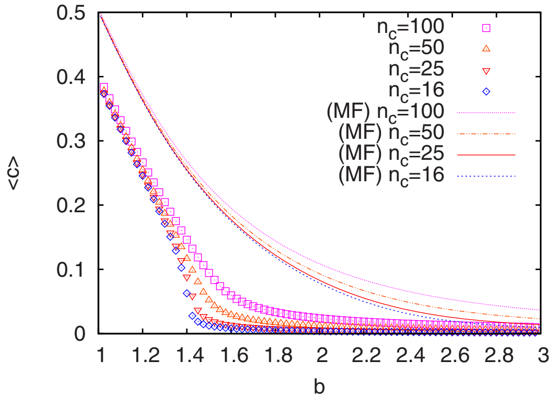

FIG. 2. (Color online) Macroscopic cooperation in a random regular graph structure for the set $\mathcal{F}$, with $k=4$, $n_F=4000$, and $\epsilon=0$. A decreasing sequence of n_C/n_F, as indicated in figure, has been used. Symbols represent numerical Monte Carlo results, and the different lines represent the mean-field predictions as given by the solution ($\dot{c}=0$) of Eq. (17).

is then ensured, in agreement with the theorem of Sec. III. In Fig. 2 we show the asymptotic value of the average cooperation c versus the temptation parameter b, as obtained from Eq. (17), for several different values of n_C/n_F, $\epsilon=0$, and $k=4$.

Within the mean field approximation, it is possible to obtain explicitly the equilibrium macro-state, i.e., the stationary probability distribution density $\mathcal{P}_l^*$, which as expected from Sec. III turns out to be of the Boltzmann type. Let us consider two different (arbitrary) strategic microstates $l=\{s_i\}$ ($i=1,\ldots,n_F$), and $l'=\{s_i'\}$, of the fluctuating set. For any pair of microstates (l, l') we define the following numbers:

$$n_{11}=\sum_i \delta_{s_i,s_i'}\delta_{s_i',1}, \tag{22}$$

$$n_{10}=\sum_i (1-\delta_{s_i,s_i'})\delta_{s_i',0}, \tag{23}$$

$$n_{00}=\sum_i \delta_{s_i,s_i'}\delta_{s_i',0}, \tag{24}$$

$$n_{01}=\sum_i (1-\delta_{s_i,s_i'})\delta_{s_i',1}, \tag{25}$$

i.e., n_{11} is the number of nodes that are cooperators in both microstates, n_{10} that of the nodes that are cooperators in l but defectors in l', etc. Using Eq. (2) it is straightforward to obtain

$$c_l-c_{l'}=\frac{1}{n_F}(n_{10}-n_{01}). \tag{26}$$

Now, let us assume that the probabilities that a node i changes strategy are independent of node i (homogeneity assumption, mean-field), and denote them by Q_{CD} (transition from defector to cooperator) and Q_{DC} (for the transition from cooperator to defector). Then we can easily see that the transition probabilities between the microstates l and l' are given by

$$\Pi_{l,l'}=(1-Q_{DC})^{n_{11}}(1-Q_{CD})^{n_{00}}Q_{DC}^{n_{01}}Q_{CD}^{n_{10}}, \tag{27}$$

$$\Pi_{l',l}=(1-Q_{DC})^{n_{11}}(1-Q_{CD})^{n_{00}}Q_{DC}^{n_{10}}Q_{CD}^{n_{01}}. \tag{28}$$

Henceforth, denoting $\exp(-\beta)=Q_{CD}/Q_{DC}$, one easily obtains the expression

$$\Pi_{l,l'}\exp(-\beta c_{l'}n_F)=\Pi_{l',l}\exp(-\beta c_l n_F), \tag{29}$$

from where the unique solution to the fixed point equation

$$\Pi_{l,l'}\mathcal{P}_{l'}^*=\mathcal{P}_l^* \tag{30}$$

is easily found to be

$$\mathcal{P}_l^*=Z^{-1}\exp(-\beta c_l n_F), \tag{31}$$

where Z is the analog of the canonical partition function

$$Z=\left[\frac{Q_{CD}+Q_{DC}}{Q_{DC}}\right]^{n_F}. \tag{32}$$

Note that Eq. (29) expresses the “detailed balance” condition, which is thus proved to be satisfied. As is well known [32], the canonical probability distribution density (31) is the unique density that maximizes the lack of information (entropy) $S=-\Sigma_l\mathcal{P}_l\ln\mathcal{P}_l$ among those (compatible) densities that share a common value for the macroscopic average of cooperation $c=\Sigma_l\mathcal{P}_l c_l$. This provides a “generalized thermodynamic” meaning to the parameter β: it is no other than the intensive entropic parameter associated to cooperation, that is, the Lagrange multiplier [33,34] associated to the restriction $c=\Sigma_l\mathcal{P}_l c_l$ on the compatible measures (canonical restricted maximization of entropy), that is,

$$\beta=\frac{\partial S}{n_F\partial c}. \tag{33}$$

The parameter β simply measures how fast the entropy of the equilibrium macrostate increases versus global cooperation variations. Its formal role is that of an analog of inverse thermodynamical temperature. Let us note that, at variance with many works in evolutionary game dynamics (see Ref. [13], and references therein) where an analog of temperature is introduced “*ad hoc*” as a parameter entering into the definition of the (stochastic) strategic updating rules, the parameter β (33) is a kind of emergent property that characterizes the equilibrium macrostate, and thus is a function of the model parameters (not a model parameter itself).

The fluctuations of the microstates cooperation c_l, namely, $(n_F)^2[\Sigma_l(\mathcal{P}_l c_l^2)-(\Sigma_l\mathcal{P}_l c_l)^2]$ are given by $n_F c(1-c)$. This is the analog of the heat capacity. The dependence on the game and network parameters b, ϵ, n_C/n_F, k of the fluctuations of cooperation is obtained by solving for the cooperation equilibrium value $\dot{c}=0$ in Eq. (17), and plotted in Fig. 3(b) for $k=4$, $\epsilon=0$, and decreasing values of the ratio n_C/n_F.

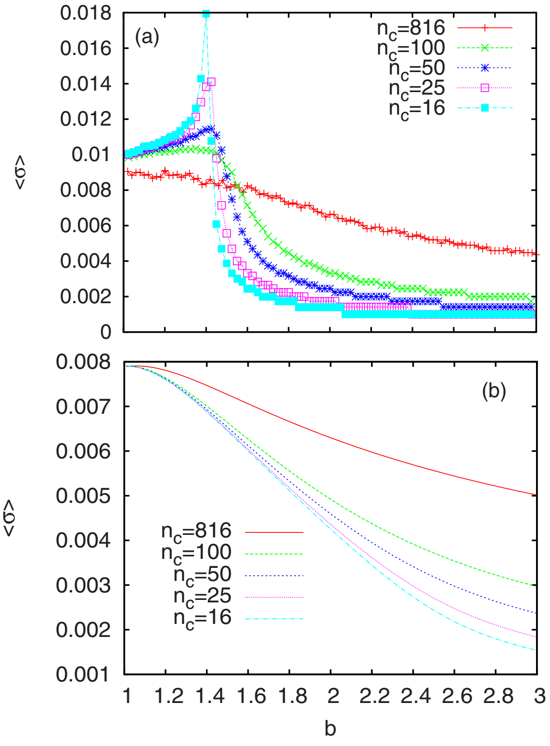

FIG. 3. (Color online) Fluctuations of cooperation in a random regular graph structure for the set $\mathcal{F}$. The upper panel (a) shows, for $k=4$, $\epsilon=0$, $n_F=4000$, and a decreasing sequence of n_C/n_F values as indicated, the fluctuations of cooperation observed in Monte Carlo simulations. The lower panel (b) shows the mean-field predictions. The mean-field approach is shown in text to be unable to predict the observed phase transition. This qualifies network reciprocity as a true "critical" social phenomenon.

2. Numerical results, and the mean-field failure

In this subsection we compare the mean-field results with those obtained from Monte Carlo simulations implementing the updating rules on the dipole model with a random regular network structure for the fluctuating set $\mathcal{F}$. In order to illustrate the Boltzmannian character of the stationary probability density $\mathcal{P}^*(l)$, we plot in Fig. 4 the numerical estimates of $\ln(\frac{P(c)}{g(c)})$, where $P(c)$ is the probability that a microstate has an average cooperation c (2), as inferred from the simulation results, and $g(c)=n_F!/[(cn_F)!(n_F-cn_F)!]$ is the degeneracy of c (i.e., the number of microstates l such that $c_l=c$). The data correspond to a random regular network structure for the component $\mathcal{F}$ with degree $k=4$, and parameter values $b=1.1$, $n_F=5000$, $n_C=500$, and $\epsilon=0$. As one can see from the perfect straight line shape of the plot, the data are fully consistent with the Boltzmann's density (31).

Though the system evolution is governed by dynamical rules (strategic updating) which "*a priori*" could be thought to lead to nonequilibrium behaviors, one finds that the asymptotic regime of the PD evolutionary dynamics in the dipole model is a true macroscopic equilibrium regime, where the formalism of generalized thermodynamics [34] applies.

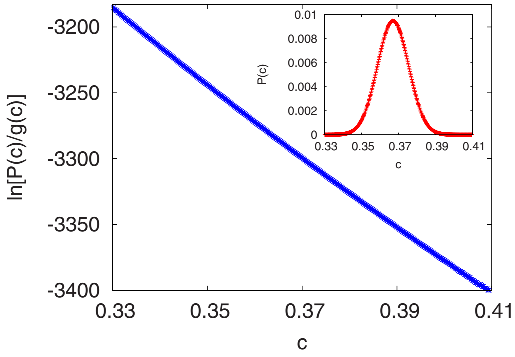

FIG. 4. (Color online) Plot of $\ln(\frac{P(c)}{g(c)})$ versus cooperation c [the inset shows $P(c)$], showing the Boltzmannian character of the stationary probability density of microstates, for a random regular network structure for the set $\mathcal{F}$. The parameter values are $b=1.1$, $n_F=5000$, $k=4$, $n_C=500$, and $\epsilon=0$. The results shown here correspond to 5×10^4 Monte Carlo steps (after a long enough transient), for each one of the 1.5×10^3 different network realizations and/or initial conditions.

The results of the asymptotic value of the average cooperation c versus the temptation to defect b are presented in Fig. 2 for (relatively small) values of n_C/n_F ranging from 0.025 down to 4×10^{-3}, but still satisfying the sufficient condition for the fixation of cooperation at BB2. The comparison with the mean-field predictions show that the mean-field approximation overestimates the cooperation value. Most notably, for very small values of n_C/n_F, the numerical results show, at about $b\simeq1.4$, a fast decay of cooperation to values close to zero (thus suggesting the existence of a phase transition), while the corresponding decay for the mean-field prediction is smooth in the whole range.

To which extent the mean-field approximation fails for low values of the parameter n_C/n_F, can be appreciated by confronting its prediction $n_Fc(1-c)$ for the fluctuations of cooperation with the results from Monte Carlo simulations. In Fig. 3(a) we see how a peak in cooperation fluctuations is revealed, when $n_C/n_F\to0$, signaling the occurrence of a phase transition between two qualitatively different equilibrium macroscopic behaviors, that correspond to low and high temptation regimes. The mean-field assumption is thus qualitatively wrong if the payoff received from $\mathcal{C}$ by big brother 2 becomes negligible versus the size n_F.

The reasons for this qualitative failure of the mean-field approximation rely on the lattice reciprocity of internal interactions, which is totally absent in the mean-field approximation. Let us remind the reader here of our remark above on the replicator-equation-type of effect of internal interactions in Eq. (8) because of the mean-field assumption. The transition signaled by the divergence of fluctuations at b^* reveals the onset of internal lattice reciprocity, a conclusion that we now substantiate (see also Appendix B below).

For $b>b^*$, say in the low-temperature (high temptation) phase, the macrostate is dominated by fast defection invasions on the relatively few nodes that are instantaneous cooperators due to sporadic interactions with big brother 2. In Appendix B we show that, in the low c and low n_C/n_F regime, the BB-imitation events in a given node are typically separated by intervals of time of about c^{-1} time units large. In those large intervals when big brother 2's influence is null, the very few and mostly isolated instantaneous cooperators are quickly invaded by defector internal neighbors. In this regime lattice reciprocity has no chance to develop, and cooperation is only weakly sustained by the sporadic influence of BB2.

On the contrary, for $b < b^*$ (high temperature, or low temptation phase) the local fluctuations of the neighbors strategic field favor the building up of clusters of cooperators that resist invasions during time intervals that are comparable to the characteristic time intervals between BB-imitation events. Under these circumstances the "extra payoff" that BB2 receives from $\mathcal{C}$ does not anymore need to be high in order to sustain high levels of cooperation. Internal lattice reciprocity enhances the probability of highly cooperative micro-states, so that the macro-states below transition differs substantially from those of the high-temptation phase. This was not captured by the mean-field approximation, for these effects require a sizable likelihood of occurrence for the local fluctuations of the strategic field, and the neglect of them is all a mean-field approach is based upon.

To summarize the discussion of the results shown in figure, a random regular structure of interactions inside $\mathcal{F}$ is enough to support lattice reciprocity mechanisms that cannot be captured by a simple mean-field approach. The onset of lattice reciprocity in the dipole model is furthermore interpreted as a "thermodynamical" phase transition, in a rigorous formal sense (divergence of the fluctuations of an equilibrium extensive parameter, the cooperation c). One is then lead to a sensible and precise formal framework where such a term as "social temperature" is not a vague metaphor, but it denotes a truly quantitative parameter, a legitimate (measurable, observable) social indicator.

V. PROSPECTIVE REMARKS

The plausibility of a thermodynamical perspective on evolutionary game dynamics studies is not a new issue, for it is somehow implicit (or at least connatural) to a body of research literature on statistical mechanics of strategic interactions [13,35]. What our simple analysis here shows is that it can sometimes be strengthened up to a formal interpretation of quantitative macroscopic social indicators as thermodynamic quantities. In the extent that it helps to understand and to quantitatively characterize the phenomenology of social and economical models, it should be recognized as a powerful theoretical perspective. Even more importantly, this perspective emphasizes the central role of quantitative (experimental, observational) studies in social sciences, and could provide, in those contexts, alternate valuable meanings to quantitative social indicators and even suggestions for new and better ones.

Any "general-physics" trained scientist recognizes that entropy reasoning is an extraordinary powerful tool for the analysis of macroscopic behavior in (material) traditional-physics systems. It turns out that some of the models (at least a bunch of interesting ones) of social phenomena are to a large extent amenable to a macroscopic description where thermodynamical concepts have proved to be essential. Of course, some notions such as, e.g., the first law of thermodynamics, could often be absent in these new contexts. However, we emphasize that the absence of energy as a variable in social models is not a shortcoming for the applicability *mutatis mutandi* of many aspects of the thermodynamical formalism to these models. A word of caution is nevertheless worth here regarding typical system sizes in controlled social experiments, where finite size effects could be hugely determinant. Also, one should not expect always social processes to be amenable to equilibrium descriptions, what makes them even more interesting from the physicists point of view.

Nowadays, it is somewhat generally accepted that physics, in general, and statistical physics, in particular, offers a powerful tool-box for problem solving in social sciences and many other areas. Recent trends in cognitive science [36] have correctly emphasized the power of the "diversity of perspectives" in problem solving, so it does not come as a surprise that adding physical perspectives to social models may sometimes pave the way to the needed breakthrough. Perhaps one should also wonder about the possibility of reverse flow in these interdisciplinary approaches to social sciences. After all, the proper use of a tool helps to its reshaping, and one could perhaps expect some kind of feedback. In other words, is there any new physics that we can learn from the study of social and economic complex systems? Only the recourse to empirical and quantitative methods in the study of social phenomena may likely give clues for sensible answers to this question.

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APPENDIX A: WHAT IF BB1 AND BB2 ARE DIRECTLY CONNECTED?

If a direct connection between big brothers is added (for the set of initial conditions specified in Sec. III, and the conditions on parameters given ibidem), one must compare their respective payoffs to see who can invade the other. One easily finds that the payoff of the defector BB1 is higher than that of the cooperator BB2 provided the following condition holds:

$$\frac{n_C}{n_F} < c(b + \epsilon - 1) - \epsilon + \frac{b - \epsilon}{n_F}, \tag{A1}$$

where c is the (instantaneous) average cooperation in $\mathcal{F}$. In this case, BB2 will be invaded with a nonzero probability. Once this eventuality occurs, no cooperator (in $\mathcal{F}$ or in $\mathcal{C}$) can later invade BB2 because all of them have lower payoffs, and fixation of defection in the whole network will occur. Note that as the average cooperation in $\mathcal{F}$ fluctuates, the condition above must be satisfied at the precise time when BB2 has chosen (by chance) to compare its payoff with BB1, and that due to the high connectivity of BB2 (which is now n_C+n_F+1) the later event occurs with a very low probability for macroscopic values of n_F. In other words, the eventual invasion of BB2 from BB1 and the subsequent fixation of defection in the whole network can take on a very long time.

If the opposite condition holds, say, if

$$\frac{n_C}{n_F} > c(b+\epsilon-1)-\epsilon+\frac{b-\epsilon}{n_F}, \tag{A2}$$

when BB1 has chosen to compare its payoff with BB2, then invasion of BB1 will occur with a nonzero probability. After this has occurred, BB1 becomes a fluctuating node (for it could be eventually invaded by an instantaneous defector in $\mathcal{F}$), but in the long term fixation of cooperation in the whole network will occur. The introduction of a direct connection between big brothers in the dipole model makes fixation of opposite strategies on them impossible, and then asymptotic fixation on the whole network of either defection or cooperation will occur, depending on the relative size n_C/n_F of components $\mathcal{C}$ and $\mathcal{F}$.

APPENDIX B: LOW *c* APPROXIMATION

In order to simplify expressions we assume hereafter $\epsilon=0$ and $k=4$, and denote $\delta=n_C/n_F$. For the case of a random regular graph structure of the fluctuating set $\mathcal{F}$, the probability $\Pi^{BB}_{C\leftarrow D}$ that an instantaneous defector node chooses to imitate big brother 2 (invasion event from BB2) is, to first order in n_F^{-1},

$$\Pi^{BB}_{C\leftarrow D}=\frac{1}{(k+2)}\frac{c+\delta}{(1+\delta)b} \tag{B1}$$

while the probability $\Pi^{BB}_{D\leftarrow C}$ of an invasion event from BB1 to an instantaneous cooperator node in $\mathcal{F}$ is, to first order in n_F^{-1},

$$\Pi^{BB}_{D\leftarrow C}=\frac{c}{(k+2)}. \tag{B2}$$

Thus, for $\delta\leqslant c$, typical intervals between invasion events from big brothers in a node are (of the order of) c^{-1} time units large. For large values of the temptation, where the value of c is expected to be very small, the dynamics is consequently dominated, for typically very large intervals of time, by internal strategic interactions. Let us analyze them.

The internal neighbors of a cooperator i are overwhelmingly likely instantaneous defectors in this "low c" regime, so that i will be quickly invaded by them. The only chance for it to resist invasion would be that its instantaneous neighborhood microstate had at least two cooperator neighbors and that $b<(3/2)$ (note that in this strategic configuration, the payoff of i is $P_i=3$ and that of its typical defector neighbors is $2b$). These neighborhood microstates (cooperative clusters) are so rare fluctuations that low values of the temptation b are necessary for their non-negligible occurrence. Provided b is below the transition value, the resilience to invasion (lattice reciprocity) of cooperative clusters enhances the likelihood of these fluctuations, which in turn reinforces the clusters resilience, and so on. This positive feedback mechanism of cooperative fluctuations enhancement is thus what triggers the transition to highly cooperative macrostates, and qualifies lattice reciprocity as a critical social phenomenon.

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